

Taylor Vortex Flow Dynamics

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THE PROBLEM:

INSTABILITIES IN FLOW BETWEEN CONCENTRIC ROTATING CYLINDERS

The general problem of flow between concentric rotating cylinders is now often called Taylor vortex flow or Couette-Taylor flow. Couette was a French scientist who in 1888 determined the viscosity of liquids from measurements of the torque for laminar flow between concentric rotating cylinders. The laminar flow state is a unique solution to the equation of motion (Navier-Stokes equation) at low cylinder rotation rates f_{cyl} : each parcel of fluid simply rotates in circles about the cylinder axis. That is, the velocity field has only one nonzero component, the azimuthal component, $v_\phi = Ar + Br$, where A and B are constants whose values are determined by the no-slip boundary conditions at the inner and outer cylinder walls, that is, $v_\phi = 2\pi f_{cyl}a$ at the inner cylinder and $v_\phi = 0$ at the outer cylinder.

The laminar base flow is a solution to the Navier-Stokes equation for *any* value of f_{cyl} . However, in a landmark paper in 1923 G.I. Taylor reported that the laminar state becomes unstable at a well-defined critical cylinder rotation rate $(f_{cyl})_c$. He obtained remarkable agreement between a theoretical analysis and a precision laboratory experiment. At the instability, time-independent toroidal-shaped vortices emerged. These “Taylor vortices” encircle the inner cylinder and are stacked along the cylinder axis.

The Navier-Stokes equation for incompressible fluids can be written in dimensionless form with only a single control parameter, the Reynolds number R , which can be defined for any system as a (characteristic length) \times (characteristic velocity) / (kinematic viscosity ν). Taylor’s work yielded the critical Reynolds number R_c for instability of flow between rotating cylinders. His theoretical and experimental results capped a half century of effort by leading scientists of the time (including Kelvin, Rayleigh, Sommerfeld, and Orr) to calculate R_c for *any* geometry. The result also provided strong evidence for the validity of the Navier-Stokes equation and for the assumption that fluid at a wall has the same tangential velocity as the wall (the “no-slip” boundary condition).

In 1944 Lev Landau proposed that, as the Reynolds number for a flow is increased, the transition to turbulence will in general occur through a sequence of instabilities, each of which would bring a new characteristic frequency into the dynamics. In 1965 Coles investigated the stability of Taylor vortices and observed that waves with a characteristic frequency spontaneously form on the vortices at a well-defined R greater than R_c . Coles conjectured that, as Landau had proposed, a sequence of such transitions would occur with increasing Reynolds number, resulting in many distinct frequencies characterizing turbulent Taylor vortex flow; however, Coles did not have the means to measure characteristic frequencies in a complex flow. The characteristic frequencies of a flow were first determined in 1975 in a

study of Taylor vortex flow by Gollub and Swinney, who found that after two successive transitions added characteristic frequencies to the dynamics, the flow became *chaotic* rather than simply multiperiodic.

In the past three decades there have been about two thousand papers reporting experimental, theoretical, and computational studies of flow between concentric cylinders. This system has become the classic prototype for investigations of instabilities, symmetry breaking, chaos, and turbulence in fluid dynamics.

HANDS-ON RESEARCH PROJECT

We will measure the characteristic frequencies for flow between rotating cylinders as a function of Reynolds number. Spatial patterns will be made visible by suspending small flat flakes ("Kalliroscope" rheoscopic fluid) into the fluid. The spatial patterns will be recorded in a computer using an inexpensive (\$100) high resolution web camera. Time sequences of the light intensity from individual pixels in the images will be Fourier transformed to obtain power spectra. Characteristic frequencies of the system will be obtained from the power spectra. The spectra will be examined to identify transitions (bifurcations) in the flow with increasing Reynolds number, and to characterize the increasingly complex dynamics that emerges with increasing rotation rate of the cylinder.

Apparatus

Inner cylinder radius $a = 3.10$ cm

Outer cylinder radius $b = 3.48$ cm

Radius ratio $= a/b = 0.891$

Gap $= d = b - a = 0.380$ cm

Inner cylinder rotation frequency f_{cyl}

Outer cylinder rotation frequency $= 0$ (in the general problem the outer cylinder as well as the inner cylinder rotates)

Annulus height H (set by annular rings at ends) $= 7.6$ cm (cylinder ht $L = 12.45$ cm)

Aspect ratio $H/d = 20.0$

Note: If the wavelength of a vortex pair were $2d$, the number of vortex pairs $H/2d$ would be 10. Typically the wavelength of a vortex pair is in the range $2d - 4d$. At a given R there can be different numbers of vorticities, which is an example of multiplicity in nonequilibrium systems. The number of different stable states (with different numbers of vortices) depends on Reynolds number.

Kinematic viscosity ν of water: At 20 degrees Celsius: 0.0100 cm²/s, at 27.5 degrees, 0.00845 cm²/s (-2.1% decrease per degree). Note: the kinematic viscosity is defined as the dynamic viscosity divided by density.

Reynolds number: The velocity scale can be taken as the tangential velocity of the inner cylinder, $(2\pi f_{cyl})a$. The length scale can be taken to be the gap between the concentric cylinders, $d = b - a$. Thus $R = (2\pi f_{cyl} a) d / \nu$. The critical Reynolds number R_c for infinitely long cylinders with radius ratio 0.89 is 125 (from a linear stability analysis), and the critical wavelength is $2.008d$.

HANDS-ON OBSERVATIONS

Instrumentation is described in *Couette-Taylor Table Top Experiment* (a separate file), which has four parts: Motor Control, Video Capture, Video Processing, and Data Analysis.

Some suggestions for explorations of the Taylor vortex flow system follow.

1. *Onset of Taylor vortex flow: instability of the laminar base flow*

At a small cylinder rotation rate, the velocity field has only one nonzero component, the azimuthal component, $v_\phi = Ar + Br$, where A and B are constants whose values are determined by the no-slip boundary conditions at the inner and outer cylinder walls, that is, $v_\phi = 2\pi f_{cyl}a$ at the inner cylinder and $v_\phi = 0$ at the outer cylinder. This laminar flow solution, where each parcel of fluid simply rotates in circular paths about the rotation axis, was known by Couette (1888) and Mallock (1888).

In 1923 Taylor found that when f_{cyl} is increased from zero, a critical value $(f_{cyl})_c$ is reached where there is a spontaneous transition from laminar flow to a flow with toroidal vortices, now called Taylor vortices. Taylor calculated $(f_{cyl})_c$ from a linear stability analysis and conducted an experiment that yielded a value of $(f_{cyl})_c$ in excellent accord with his prediction. The transition from laminar flow to Taylor vortex flow, is an example of a *pitchfork bifurcation*, that is, a transition from one time-independent state to another with different symmetry. At a pitchfork bifurcation the amplitude of the symmetry breaking mode grows as the square root of the distance above the instability.

Suggestions for studying the bifurcation to Taylor vortex flow:

1.1 Learn how to set motor speed and acceleration rate (see Instrumentation: Motor Control).

1.1 Increase the cylinder rotation frequency from rest until Taylor vortices are observed to form. Do the vortices emerge in the middle or at the ends, or everywhere simultaneously? Why?

1.2 Is there hysteresis in the transition from laminar flow to Taylor vortex flow?

1.3 How does your R_c compare with the result of a linear stability analysis?

1.4 What is the axial wavelength at R_c ? How does it compare with the linear stability analysis?

2. *Transition to wavy vortex flow*

2.1 Increase R slowly upward from R_c until traveling waves appear on the Taylor vortices. What is the value of this second critical Reynolds number?

2.2 Is this *Hopf bifurcation* (transition from a time-independent to a periodic state) hysteretic? How does the instability Reynolds number depend on the acceleration rate and the direction of the acceleration?

2.3 Estimate by visual observation how many azimuthal waves are present.

3. *Multiplicity:*

A nonlinear dynamical system can have multiple stable states (multiple basins of attraction in phase space) at a given value of the relevant control parameters. In some cases a system may have attractors that are time-independent (a fixed point in phase space), periodic (limit cycle), doubly periodic (2-torus), multiperiodic (n-torus), and chaotic, all occurring at the same value of control parameter. Each attractor has its own basin of attraction; the boundaries between the different basins are often fractal.

Multiplicity in the Taylor system: different numbers m of azimuthal waves and different numbers N of Taylor vortices can form for a given R . Examine this multiplicity.

3.1 If R is increased from zero to a value in the range approximately $3R_c - 8R_c$, the number of waves can vary from 4 to 8, depending on the Reynolds number history.

3.2. The axial wavelength at some R can vary from around $2d$ to $4d$, depending on the Reynolds number history. Use different acceleration rates to obtain different numbers of vortices.

3.3 Here is a trick to reach a state with a longer wavelength. With a fixed R and N vortices, quickly drain part of the fluid (say one-third) from the annulus. The vortices near the bottom will disappear down the drain. Then if the annulus is refilled very slowly, the vortices will grow in size, increasing the axial wavelength.

Alternatively, a state with a shorter wavelength can be achieved by very slowly draining fluid and then rapidly replacing the lost fluid.

4. *Make movies and compute power spectra* (see Instrumentation)

The first two of the following considerations are essential. The next two should be skipped in a first study of wavy vortex power spectra.

4.1 *Frame rate for movie:* By the Nyquist theorem, the maximum frequency of a spectrum is one-half the sampling rate. Since the characteristic frequency f_1 for wavy Taylor vortex patterns is of order f_{cyl} , the movie frame rate should be at least $2f_{cyl}$. In order to increase confidence in the interpretation of the spectra; the spectra should include at least the second harmonic; hence choose a frame rate $4f_{cyl}$ to $6f_{cyl}$.

4.2 *Movie duration:* The width of a peak [full width at half maximum (FWHM)] in a power spectrum corresponding to a sinusoidal signal is approximately $1/(2T)$, where T is the duration of the data set. If this width is to be 1% of the component frequency f_1 , then the duration of the data set should be at least $50/f_1$, or about $50/f_{cyl}$ since $f_1 \sim f_{cyl}$.

4.3 *Aliasing:* A spectral component f^* that is undersampled will fold back into the spectrum giving a component at $2f_{NYQUIST} - f^*$. If the highest frequency component in a spectrum is $2f_1$, then $3f_1$ will be outside of the spectral range and will hence be undersampled. Look for this component then at $2f_{NYQUIST} - 3f_1$.

4.4 *Signal to noise:* As usual in signal analysis, there is a trade-off between signal-to-noise and resolution. Noise can be reduced by averaging many spectra. For example, a long time series (e.g., duration $1000/f_1$ or $10000/f_1$) can be split into many consecutive pieces of equal size that are separately Fourier transformed to obtain power spectra. The signal to noise of the averaged spectra will have an improved signal to noise.

5. *Power spectral analysis of wavy vortex flow*

Power spectra: plot $\log(\text{power})$ [because of the very large dynamic range of the power] vs f/f_{cyl} [because f_1/f_{cyl} changes quite slowly with increasing R].

5.1 Increase R slowly upward from R_c until waves appear on the Taylor vortices. What is the characteristic frequency f_1 of these waves relative to the cylinder frequency? Measure the frequency at several R (say increasing R by 10% each time), and determine how rapidly the characteristic frequency changes.

5.2 How many waves are there at the onset of instability? What is the wave speed?

5.3 Measure the spectral resolution, the half-width at half-maximum of f_1 (with R in the range $1.5R_c$ to $4R_c$). The width should be $\sim 1\%$ of f_1 for a data set of duration $\sim 50/f_1$.

5.4 A bifurcation from a time-independent system to a periodic state is called a *Hopf bifurcation*. Beyond the onset of the bifurcation the amplitude of the oscillatory mode generically increases as the square root of the distance in control parameter (here R) above the bifurcation point. You can determine the amplitude of the oscillatory mode by integrating the area under the spectral peaks.

6. *Modulated wavy vortex flow* (flow with two incommensurate characteristic frequencies)

With an increase in R for a wavy vortex state, at some larger R another characteristic frequency f_2 will appear in the power spectrum. Visual observations will reveal that the Taylor vortex pattern is modulated – that is, the flow pattern is time dependent even in the co-rotating frame of the original wavy vortices. The frequency f_2 corresponds to an azimuthal traveling wave similar to the wavy vortices; the angular speed of the modulated waves is about $0.44f_{cyl}$, and typically the number m_2 of the secondary waves around the annulus is 4, resulting in $f_2 \approx 4 \times 0.44f_{cyl} = 1.76 f_{cyl}$.

6.1 Start with a wavy vortex flow state with four waves at about $8R_c$. Increase R slowly until a second characteristic frequency f_2 appears in the power spectrum.

6.2 Observe the affect of the modulation on the waves in the movies.

6.3 Commensurate and incommensurate frequencies can never be distinguished experimentally, but it is possible to obtain strong evidence for incommensurability. There will probably be insufficient time to obtain evidence of incommensurability, but can you suggest how to obtain such evidence?

7. *Phase space portraits.*

An n -dimensional phase space for a dynamical system can be constructed from vectors obtained from a time series for a single variable $x(t)$ by using the method of time delays, where a point in the space is given by $X(t) = \{x(t), x(t-\tau), x(t-2\tau), \dots, x(t-(n-1)\tau)\}$. The phase portrait characterizing the dynamics is given by the trajectory $X(t)$ in the n -dimensional phase space. For a d -dimensional attractor an embedding in the phase space is assured if $n > 2d$ by the Takens embedding theorem (1981), which is a generalization of the Whitney embedding theorem (1936); see the Roux, Simoyi, & Swinney, *Physica D* **8**, 257 (1983) [pdf file in the folder of references].

7.1 Construct the phase space attractors for a flow with one fundamental frequency (limit cycle attractor) and for a flow with two incommensurate characteristic frequencies (a 2-torus attractor).

7.2 Construct the phase space portrait for a case where the power spectrum includes broadband noise in addition to sharp characteristic frequencies.

8. *Chaos, Strange Attractors, Lyapunov exponents, fractal dimension, turbulence.*

The largest Lyapunov exponent characterizes the exponential rate of separation in phase space of two points that were initially infinitesimally separated. A system is chaotic if the largest Lyapunov exponent is positive. In practice the fractal dimension and Lyapunov exponent spectrum can be determined from laboratory observations only for systems whose dynamics is characterized by a low dimensional phase space attractor (say dimension up to around 6). The number of data points needed increases roughly as $\sim 10^d$. However, the concept of Lyapunov exponents can be very useful even for strongly turbulent flows that have coherent structures (jets and vortices).

A turbulent flow is characterized not only by sensitive dependence on initial conditions (the defining property of chaos), but also by a large range in spatial and temporal scales. The scaling of turbulence in concentric cylinder systems has been studied for Reynolds numbers up to $\sim 10^6$. It is difficult to go to higher R in controlled laboratory experiments because of Joule heating due to the stirring by differential rotation of the cylinders; the power dissipation with water as a working fluid can easily be \sim kilowatts at $R \sim 10^6$.

ENJOY!

COUETTE-TAYLOR TABLE TOP EXPERIMENT

SOFTWARE AND HARDWARE OPERATION GUIDE

Motor Control –

a) Stepper Motor

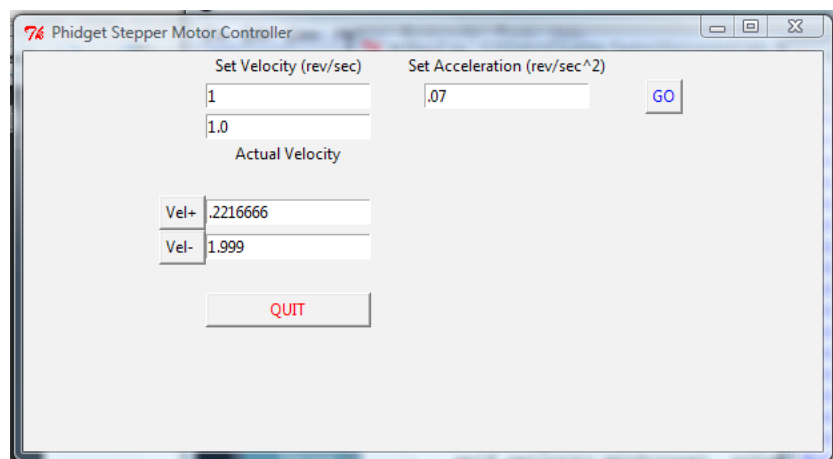
i) A stepper motor (Lin Engineering) provides control of the cylinder rotation rate. The stepper motor works by precisely controlling the current to its four motor windings. This allows the motor to rotate in small increments. The increment is based on the motor resolution. The Lin Engineering motor being used has a resolution of 400 steps per revolution. A stepper motor can start, stop and reverse directions very quickly and accurately. This experiment does not require a stepper motor, which costs as little as \$10 to \$20 US. The motor being used cost about \$50, but a less expensive stepper motor would work as well. An inexpensive DC motor is another alternative. The primary advantage of a stepper motor is that no feedback is required to achieve precise control of the rotation and acceleration rates, whereas with a DC motor, such feedback would be necessary.

b) Phidgets Step Motor Controller and Software

i) The Phidget step motor controller provides the current to the motor windings as described above. It is a low-cost controller that costs \$80 US whereas some controllers cost \$400-\$500 US or more. It does not come with a prebuilt software interface, but does include well-coded software libraries allowing the user to create an application in a variety of languages, including Matlab. We used Python to create a GUI for motor control because it is open source software and has some programming advantages over Matlab.

ii) The Phidget controller has a velocity range from 0 to 5 rev/sec and an acceleration range of .625 to 2 rev/sec². The acceleration lower limit is far too fast for the Couette-Taylor system, where we may want acceleration values as low as .001 rev/sec² when precisely determining the onset of an instability. Therefore, a computer algorithm was written that reads the current velocity, the desired velocity and the desired acceleration. The controller then changes the velocity in small increments inside of a while loop to provide the desired linear acceleration profile

Figure 1:
Phidget Motor
Controller GUI
used for motor
control.



iii) For the experiment, we will double click a file named Stepper_GUI.py, located on the desktop. This file will automatically launch the Windows command prompt, but the file can be run using a Python shell program. (Python is available at www.python.org.) The GUI is entitled Phidget Motor Controller and once launched, attaches to the motor and gives feedback on the motor status. If the controller does not find the motor, it will send back an error message at the command line that it could not attach.

iii) There are three main fields in the Phidget Motor Controller GUI labeled Velocity, Acceleration and Current Velocity. The first two are used to specify the desired velocity in

rev/sec and acceleration in rev/sec^2 . The controller will execute the velocity when the “GO” button is pressed, and the button will turn green indicating that the motor is accelerating. The “GO” button will turn grey again when the acceleration is finished.

The third field is labeled Current Velocity and is updated once the motor is finished accelerating from its original to final velocity. This provides a check on the computer algorithm and gives additional notification that the motor has finished accelerating. The velocity can not be changed once the GO button is pressed until it has reached the value specified in the velocity field.

- iv) The velocity should not be set larger than 5 rev/sec. No harm will be done because the motor will simply reach its maximum velocity and stop accelerating, but the user will have to wait until the computer algorithm is finished before resetting the velocity and it could take some time to reach before this happens.
- v) The other two fields next to the “Vel+” and “Vel-” buttons are for specifying a small velocity increment that the user would like to make by pressing the respective button. The value in the increment fields can be anything but the velocity change will happen at the minimum acceleration rate provided by the controller, $.625 \text{ rev/sec}^2$, so the change should typically be only a small amount.

VIDEO CAPTURE – Logitech Quickcam Pro 9000 Webcam and VirtualDub software.

- a) Video Capture – The data will be acquired using a Logitech webcam and a freeware program called VirtualDub. This software will be used for acquiring video and can be used for post processing video, though we won’t do that in our experiment.

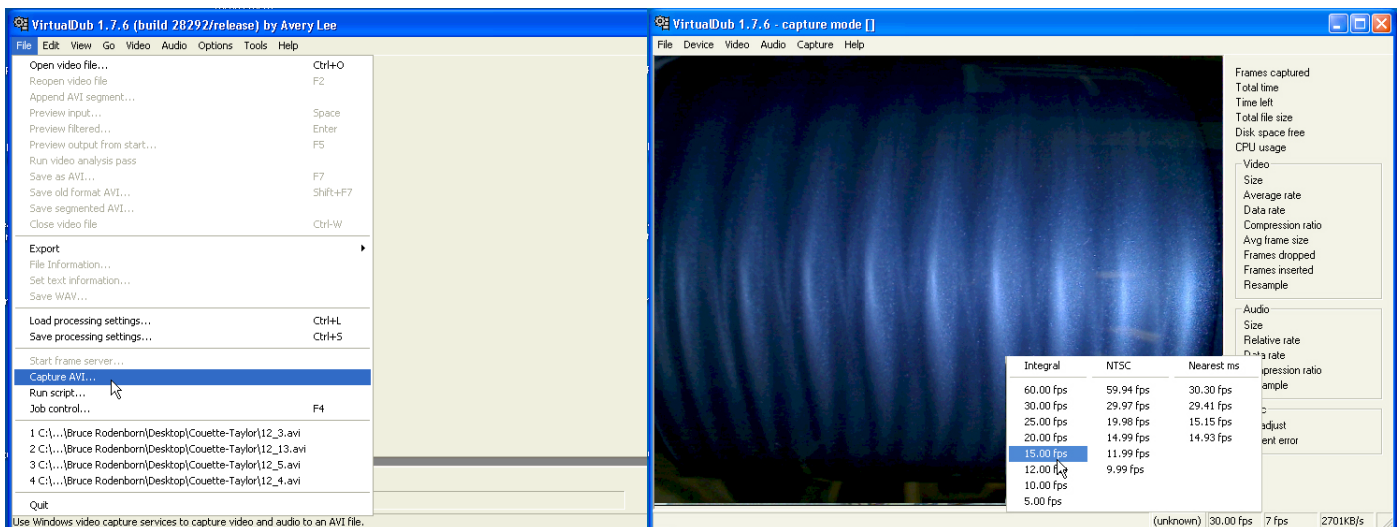
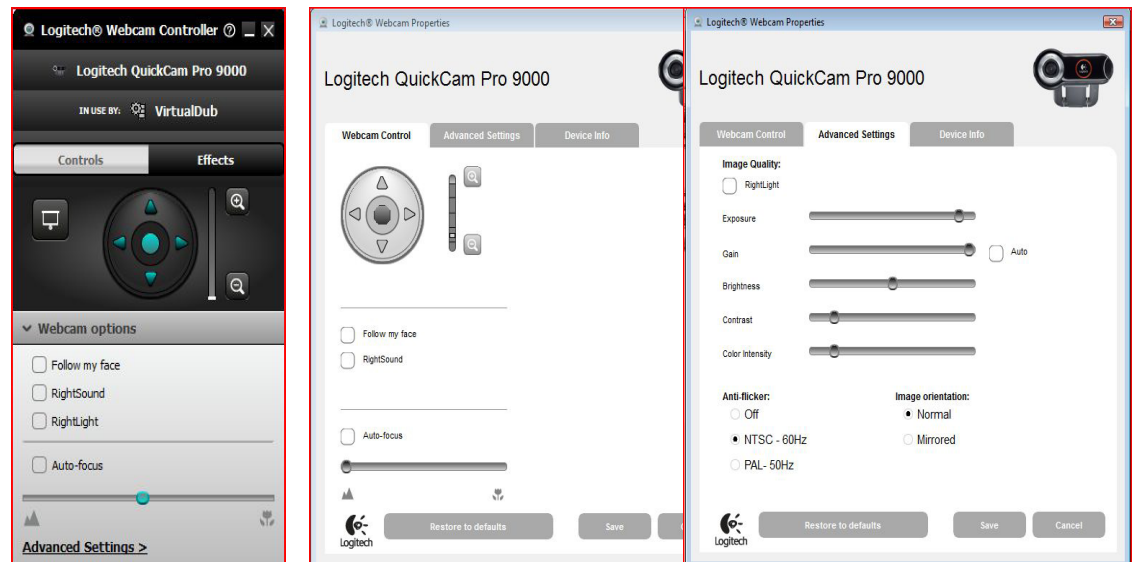


Figure 3 Virtual Dub software: a. setting a capture file, and b. setting the frame

- i) Launch VirtualDub from the shortcut on the Taskbar and then from the File menu, select “Capture AVI...” as in Figure 3 a. A live preview appears, as in in Figure 3 b.
- ii) The Logitech Quickcam application, , shown in Figure 4 above, will also launch because VirtualDub is using this driver to control the camera.
- iii) VirtualDub will record the video file directly to the hard drive and requires the user to specify the name of the AVI file to create. In the File menu, select “Set Capture File ...” and name the capture file so that it can be identified later, and save it in Desktop>Couette-Taylor_Videos.
- iv) Camera Settings - VirtualDub provides many different kinds of control of the video, but for our purposes, the main controls will be the Frame rate, Audio, Focus, Shutter Speed, Gain.
 - (a) Frame Rate – the frame rate is listed in the lower right-hand corner of the window. Click on it and a menu will pop up allowing the frame rate to specified. The allowed values are 5, 10, 12, 15, 20, 25 and 30 frames per second, as shown in Figure 3 b. Select a frame rate that is three to five times the rotation period (see Section 4 of Hands-on Observations).

- (b) Audio – To reduce the data transfer rate and the total amount of data captured, disable the audio capture by deselecting “Enable audio capture” in the Audio menu.
- (c) Focus – Though VirtualDub does provide control of this and the other remaining parameters, it is actually easier to use Logitech’s software for this purpose. In the Logitech Quick application, there is an icon that looks like two gears; click on this and the Webcam Settings will appear as shown in the image below. There is a link at the bottom of the window for “Advanced Settings.” The first looks like a webcam. Select it and the control for the camera focus appears as in Figure 4b. Adjust the slider so that the video preview in

Figure 4:
a. Logitech Quickcam software panel.
b. setting focus.
c. setting exposure and gain.



VirtualDub is in focus.

- (d) Exposure Time and Gain – Click on the image of an eye (see Figure 4c), and the controls for the Exposure and the Gain appear. A shorter exposure time is desirable for high Reynolds numbers, but the cost is total light captured, so the video image is much dimmer. Improve the lighting or boost the gain as is necessary to get a good image.
- v) Initiate the Video Capture – Go back to VirtualDub, and in the Capture menu select “Start Capture” which starts the video recording. VirtualDub provides statistics on the number of frames captured, the actual frame rate, and other information. If the actual frame rate is varying significantly, then the data won’t be usable as the time separation between samples is not consistent. If necessary, lower the frame rate to achieve a constant frame rate.
- vi) Stop the Capture – Hit the ESC key or use the “Capture” menu to stop the video capture when enough frames are recorded. Typically runs of 512 or even 256 frames are sufficient for quick determination of the frequencies present in the current state, but longer runs are necessary for detailed analysis. See the Hands-on Observations discussion of considerations in choosing frame rate.

DATA ANALYSIS - Matlab

- a) A Matlab script with an accompanying GUI has been created for processing the data. Matlab is a useful program that is particularly good at working with Matrices, but is very expensive and requires yearly licensing fees. However Matlab was originally created from open source software so there are several competing freeware applications out there such as SciLab and Octave.
- b) Launch the GUI by first opening the Matlab .m file located in the Couette-Taylor folder on the Desktop. Feel free to look at the code if it is interesting (see Figure 6a below). To launch the GUI, run the script by pressing the fn and the F5 key at the lower left and top of the keyboard respectively.
- c) A window entitled, “Couette-Taylor Power Spectral Density” will launch (Figure 6b). Using this GUI will mostly be self-explanatory, and will return the Power Spectral Density of the pixel data recorded in the video. Enter the appropriate values in the fields provided. Below is a description of

the fields:

- i) Number of Vortices – record state observed
- ii) Rotation Speed – cylinder speed in rev/sec.
- iii) Frames per Second – frame rate specified in VirtualDub for the capture
- iv) Total Number of Frames – this may be less than the total number of frames captured and exported previously, but should be a number that is 2^n (e.g. 256, 512, 1024, etc.) because they will be used for Fourier transformations.
- v) Frame Read Frequency – Sometimes it is desirable to undersample the video sequence. For example, record a video taken at 30 frames per second (fps) and analyze the data at 15 fps. The Frame Read Frequency field sets the sample rate at which frames are chosen, e.g. enter a two and every other frame will be used. Typically, though, every frame is sampled.
- d) Click the button “Get Movie Data” and a navigation window will open. Navigate to and double click on the video file recorded using Virtual Dub. The first movie frame will be displayed in the “Video

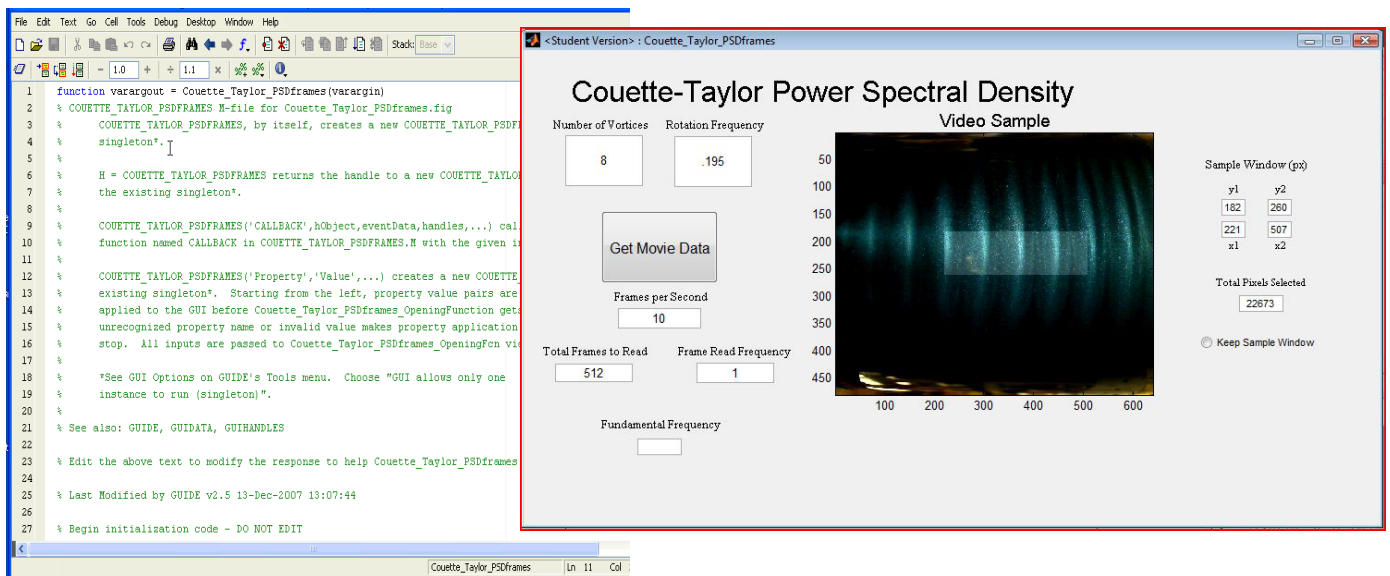


Figure 6: a. Matlab source code, and b. Power Spectral Density GUI

Sample” window of the GUI.

- e) Select Image Area - Because the entire image area contains over 307,200 pixels, analyzing every pixel is too time consuming, so the program has the user select a smaller area of the image for analysis. The Matlab script waits for the user to click two points in the “Video Sample” window. These two mouse clicks define the upper left and lower right corners of the image area to be analyzed. Generally, select an area with good lighting that crosses several vortex boundaries. Try different regions to see what works best. The absolute pixel numbers selected are displayed in the fields under “Sample Window (px)”.
- f) Keep the Sample Window – The selected sample window can be kept by pressing the radio button. This is useful for direct comparison of different states so that the exact same region is analyzed. Otherwise a different window must be selected each time.
- g) The selected area will be highlighted in the sample, and the program reads these same pixels from each frame. The program then computes the average power spectral density of each of these pixel signals. Progress of the image read and power spectral density is given in the “Calculation Update” field.
- h) Results – the program returns several tools for analyzing the spectrum.
 - i) The first is a plot of the log of the average power spectral density versus frequency in units of cylinder frequency, as in Figure 7a.
 - ii) The dominant frequency component is returned in the field “Fundamental Frequency” in the Matlab GUI. It is calculated as the first moment of the dominant peak in the spectrum.
 - iii) Using this fundamental frequency, the program time shifts the signals from the different

pixels and sums them to give an averaged pixel signal in a second plot, as shown in Figure 7b.

iv) The averaged pixel signal is used to provide a dynamical variable for the system. The program plots the $\text{Signal}(t + \Delta t)$ vs. $\text{Signal}(t)$, where Δt is $1/3$ of the fundamental period from the power spectral density, providing a two-dimensional phase-space plot as shown in Figure 7c.

Figure 7 Matlab analysis:
a. power spectral density,
b. average pixel signal,
c. phase space plot

